Izračunati integral $I = \int_1^{\sqrt{3}} \frac{x^5+1}{x^6+x^4}\,dx$.

Rj.

$$x^6+x^4 = x^4(x^2+1)$$

$$\frac{x^5+1}{x^4(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{Ex+F}{x^2+1} \quad / \cdot x^4(x^2+1)$$

$$x^5+1 = A(x^5+x^3) + B(x^4+x^2) + C(x^3+x) + D(x^2+1) + Ex^5 + Fx^4$$

$$\begin{array}{lllllll}
x^5: & A & & & & +E & & = 1 \\
x^4: & & B & & & & +F & = 0 \\
x^3: & A & & +C & & & & = 0 \\
x^2: & & B & & +D & & & = 0 \\
x: & & & C & & & & = 0 \\
x^0: & & & & D & & & = 1
\end{array}$$

$D=1$
$C=0$
$B=-1$
$A=0$
$E=1$
$F=1$

x^{-2}

x^{-4}

$d(x^2+1) = 2x\,dx$

$$\frac{x^5+1}{x^6+x^4} = \frac{-1}{x^2} + \frac{1}{x^4} + \frac{x+1}{x^2+1}$$

$$\int_1^{\sqrt{3}} \frac{x^5+1}{x^6+x^4}\,dx = -\int_1^{\sqrt{3}} \frac{dx}{x^2} + \int_1^{\sqrt{3}} \frac{dx}{x^4} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx + \int_1^{\sqrt{3}} \frac{dx}{x^2+1}$$

$$= \frac{1}{x}\Big|_1^{\sqrt{3}} + \frac{x^{-3}}{-3}\Big|_1^{\sqrt{3}} + \frac{1}{2}\ln|x^2+1|\Big|_1^{\sqrt{3}} + \operatorname{arctg} x\Big|_1^{\sqrt{3}}$$

$$= \ldots = \frac{\pi}{12} + \frac{1}{2}\ln 2 + \frac{8\sqrt{3}}{27} - \frac{2}{3}$$

traženo rješenje

Naći ekstreme f-je $z = x^3 + 4x^2y + xy^2 - 12xy - 3y^2$.

Rj.

$$\frac{\partial z}{\partial x} = 3x^2 + 8xy + y^2 - 12y$$

$$\frac{\partial z}{\partial y} = 4x^2 + 2xy - 12x - 6y$$

$$3x^2+8xy+y^2-12y=0$$
$$4x^2+2xy-12x-6y=0$$

$$3x^2+8xy+y^2-12y=0$$
$$4x(x-3)+2y(x-3)=0$$

$$3x^2+8xy+y^2-12y=0$$
$$(4x+2y)(x-3)=0$$

$$3x^2+8xy+y^2-12y=0$$
$4x+2y=0$ ili $x-3=0$

$y=-2x$ ili $x=3$

a) za $x=3$ imamo

$$3\cdot 9+8\cdot 3y+y^2-12y=0$$
$$y^2+12y+27=0$$
$$D=144-108$$
$$y_1=-9,\quad y_2=-3$$

b) $y=-2x$

$$3x^2+8x\cdot(-2x)+(-2x)^2-12(-2x)=0$$
$$3x^2-16x^2+4x^2+24x=0$$
$$-9x^2+24x=0$$
$$-3x(3x-8)=0$$
$$x_1=0,\quad x_2=\frac{8}{3}$$

Stacionarne tačke su $M_1(3,-9)$, $M_2(3,-3)$, $M_3(0,0)$, i $M_4\left(\frac{8}{3}, -\frac{16}{3}\right)$.

$$\frac{\partial^2 z}{\partial x^2} = 6x+8y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 8x+2y-12$$

$$\frac{\partial^2 z}{\partial y^2} = 2x-6$$

Za $M_1(3,-9)$

$$A=-54,\quad B=-6,\quad C=0$$

$$D=\begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC-B^2 = -36<0$$

F-ja u tački M_1 nema ekstrem.

Za $M_2(3,-3)$

$A=-6,\ B=6,\ C=0$

$D=AC-B^2=-36<0$

F-ja u tački $M_2(3,-3)$ nema ekstrem.

Za $M_3(0,0)$

$A=0,\ B=-12,\ C=-6$

$D=AC-B^2=-144<0$

F-ja u tački $M_3(0,0)$ nema ekstrem

Za $M_4\left(\frac{8}{3},-\frac{16}{3}\right)$

$A=-\frac{80}{3},\ B=-\frac{4}{3},\ C=-\frac{2}{3}$

$D=AC-B^2=16>0$

F-ja u tački $M_4\left(\frac{8}{3},-\frac{16}{3}\right)$ ima ekstrem.

$A<0 \Rightarrow$ f-ja ima maksimum

$z_{MAX}\left(\frac{8}{3},-\frac{16}{3}\right)=\ldots=\frac{256}{9}$

(#) Neka je A tačka u kojoj prava $2x-\sqrt{5}y-1=0$ siječe y-osu, a B tačka u kojoj data prava siječe x-osu. Izračunati krivolinijski integral prve vrste $\int_C \frac{ds}{\sqrt{x^2+y^2+1}}$, ako je c odsječak date prave između tačaka A i B.

Rj.

Prisjetimo se:

Ako je kriva c data u parametarskom obliku

$$c: \begin{cases} x=\eta(t) \\ y=\mu(t) \\ t_1 \le t \le t_2 \end{cases}$$

tada se krivolinijski integral prve vrste $\int_C f(x,y)\,ds$ računa po formuli

$$\int_C f(x,y)\,ds = \int_{t_1}^{t_2} f(\eta(t),\mu(t))\sqrt{(\eta'(t))^2+(\mu'(t))^2}\,dt$$

Pronađimo presječne tačke date prave sa osama

$$2x-\sqrt{5}y-1=0$$

$$2x-\sqrt{5}y=1$$

$$\frac{x}{\frac{1}{2}}+\frac{y}{-\frac{1}{\sqrt{5}}}=0$$

Tražene tačke A i B su

$$A\left(0; -\frac{1}{\sqrt{5}}\right)$$

$$B\left(\frac{1}{2}; 0\right)$$

Odredimo jednačinu prave kroz tačke A i B koristeći formulu

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$A(\overset{x_1}{0};\ \overset{y_1}{-\frac{1}{\sqrt{5}}})$

$B(\overset{x_2}{\frac{1}{2}};\ \overset{y_2}{0})$

$$\frac{x}{\frac{1}{2}}=\frac{y+\frac{1}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}\quad (=t)$$

$$x=\frac{\sqrt{5}}{2}y+\frac{1}{2}$$

$$\frac{1}{\sqrt{5}}x=\frac{1}{2}\left(y+\frac{1}{\sqrt{5}}\right)\quad /\cdot\sqrt{5}$$

$$\frac{\sqrt{5}}{2}y=x-\frac{1}{2}\quad /\cdot\frac{2}{\sqrt{5}}$$

$$x=\frac{1}{2}t$$

$$y+\frac{1}{\sqrt{5}}=\frac{1}{\sqrt{5}}t$$

$\Rightarrow$ Parametarski oblik duži c je

$$c:\begin{cases} x=\frac{1}{2}t \\ y=\frac{1}{\sqrt{5}}t-\frac{1}{\sqrt{5}} \\ 0\le t\le 1\end{cases}\Rightarrow$$

$$\dot{x}=\frac{1}{2}$$

$$\dot{y}=\frac{1}{\sqrt{5}}$$

$$\Rightarrow \dot{x}^2+\dot{y}^2=\frac{1}{4}+\frac{1}{5}=$$

$$=\frac{5+4}{20}=\frac{9}{20}$$

$$\sqrt{20}=\sqrt{5\cdot 4}=2\sqrt{5}$$

$$x^2+y^2+1=\left(\frac{1}{2}t\right)^2+\left(\frac{1}{\sqrt{5}}t-\frac{1}{\sqrt{5}}\right)^2+1\overset{\text{ZA VJEŽBU}}{=}\ldots=\frac{9}{20}t^2-\frac{2}{5}t+\frac{6}{5}$$

$$\sqrt{x^2+y^2+1}=\sqrt{\frac{9}{20}}\cdot\sqrt{t^2-\frac{8t}{9}+\frac{8}{3}}=\frac{3}{2\sqrt{5}}\sqrt{\left(t-\frac{4}{9}\right)^2+\frac{200}{81}}$$

$$\int_c\frac{ds}{\sqrt{x^2+y^2+1}}=\frac{3}{2\sqrt{5}}\cdot\frac{2\sqrt{5}}{3}\int_0^1\frac{dt}{\sqrt{\left(t-\frac{4}{9}\right)^2+\frac{200}{81}}}=\left|\begin{array}{ll} t-\frac{4}{9}=\frac{10\sqrt{2}}{9}s & t=0\Rightarrow\ldots \\ \left(t-\frac{4}{9}\right)^2=\frac{200}{81}s^2 & t=1\Rightarrow\ldots\end{array}\right|$$

$$\overset{\text{ZA VJEŽBU}}{=}\ldots=\ln\left(\frac{3\sqrt{6}}{5}+\frac{2}{5}\right)$$

Izračunati fluks vektorskog polja

$$\vec{v} = (x, -y^2, x^2+z^2-1)$$

po unutrašnjoj strani sfere $x^2+y^2+z^2=1$.

Rj.

Prisjetimo se

Fluks vektorskog polja se računa po formuli

$$\Phi = \iint_S v_x \, dy\, dz + v_y \, dx\, dz + v_z \, dx\, dy$$

U našem slučaju

$$\Phi = \iint_S x \, dy\, dz - y^2 \, dx\, dz + (x^2+z^2-1) \, dx\, dy$$

gdje je S unutrašnja strana sfere sa centrom u $C(0,0,0)$ poluprečnika 1.

Ako iskoristimo formulu Gauss-Ostrogradski imamo

$$\Phi = \iiint_\Omega (1-2y+2z) \, dx\, dy\, dz =$$

Ω je unutrašnjost date sfere - ako uvedemo sferne koordinate imamo

$x = \rho \sin\varphi \cos\alpha$

$y = \rho \sin\varphi \sin\alpha$

$z = \rho \cos\varphi$

$dx\, dy\, dz = \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\alpha$

$$\Omega \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le \pi \\ 0 \le \alpha \le 2\pi \end{cases}$$

$$= \iiint_{\Omega'} (1 - 2\rho \sin\varphi \sin\alpha + 2\rho\cos\varphi)\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\alpha =$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{1} \left(\rho^2 \sin\varphi - 2\rho^3 \sin^2\varphi \sin\theta + 2\rho^3 \sin\varphi \cos\varphi\right) d\rho$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{1}{3}\rho^3\Big|_0^1 \sin\varphi - 2\cdot\frac{1}{4}\rho^4\Big|_0^1 \sin^2\varphi \sin\theta + 2\cdot\frac{1}{4}\rho^4\Big|_0^1 \sin\varphi\cos\varphi\right) d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{1}{3}\sin\varphi - \frac{1}{4}(1-\cos 2\varphi)\sin\theta + \underbrace{\frac{1}{2}\sin\varphi\cos\varphi \Big) d\varphi}_{-\frac{1}{2}\cos\varphi \, d(\cos\varphi)}$$

$$= \int_0^{2\pi} \left(-\frac{1}{3}\underbrace{\cos\varphi\Big|_0^{\pi}}_{-1-1} - \frac{1}{4}\sin\theta\Big(\underbrace{\varphi\Big|_0^{\pi}}_{\pi} - \frac{1}{2}\underbrace{\sin 2\varphi\Big|_0^{\pi}}_{0-0}\Big) - \frac{1}{2}\cdot\frac{1}{2}\underbrace{\cos^2\varphi\Big|_0^{\pi}}_{1-1=0}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3} - \frac{1}{4}\pi \sin\theta\right) d\theta = \frac{2}{3}\theta\Big|_0^{2\pi} + \frac{1}{4}\pi \underbrace{\cos\theta\Big|_0^{2\pi}}_{1-1} =$$

$$= \frac{4\pi}{3}$$

$$1 = \sin^2\varphi + \cos^2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$1 - \cos 2\varphi = 2\sin^2\varphi$$

$$\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$$